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Action, mass and entropy of Schwarzschild-de Sitter black holes and the de Sitter/CFT correspondence

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Abstract: We investigate a recent proposal for defining a conserved mass in asymptotically de Sitter spacetimes that is based on a conjectured holographic duality between such spacetimes and euclidean conformal field theory. We show that an algorithm for deriving such terms in asymptotically anti de Sitter spacetimes has an asymptotically de Sitter counterpart, and derive the explicit form for such terms up to 9 dimensions. We show that divergences of the on-shell action for de Sitter spacetime are removed in any dimension in inflationary coordinates, but in covering coordinates a linear divergence remains in odd dimensions that cannot be cancelled by local terms that are polynomial in boundary curvature invariants. We show that the class of Schwarzschild-de Sitter black holes up to 9 dimensions has finite action and conserved mass, and construct a definition of entropy outside the cosmological horizon by generalizing the Gibbs-Duhem relation in asymptotically dS spacetimes. The entropy is agreement with that obtained from CFT methods in $d = 2$. In general our results provide further supporting evidence for a dS/CFT correspondence, although some important interpretive problems remain.

Keywords: AdS-CFT Correspondence, Classical Theories of Gravity, Black Holes.

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1. Introduction

It is generally believed that the definition of a conserved charge in a gravitational spacetime that is asymptotically de Sitter (dS) is not well defined. The reason is that such spacetimes do not have spatial infinity the way that their asymptotically flat or asymptotically anti de Sitter (AdS) counterparts do. Moreover one cannot define a timelike Killing vector in global de Sitter spacetime. In fact, there is a timelike Killing vector field inside the cosmological horizon that becomes spacelike outside the cosmological horizon. For this reason, the physical meaning of the Abbott-Deser energy outside the cosmological horizon of dS spacetime is not clear and to construct the energy, one could use the conformal Killing vector [1].

Recently a novel prescription was proposed for computing conserved charges (and associated boundary stress tensors) of asymptotically dS spacetimes from data at early or late time infinity [2]. The method is analogous to the Brown-York prescription in asymptotically AdS spacetimes [3, 4, 5], and yields suggestive information about the stress tensor and conserved charges of the hypothetical dual euclidean conformal field theory (CFT) on the spacelike boundary of the asymptotically dS spacetime, providing intriguing evidence for a holographic dual to dS spacetime that is similar to the AdS/CFT correspondence. Such a similarity also is observed in the computation of the conformal anomaly of dual euclidean conformal field theory [6]. The specific prescription in ref. [2] (which has been employed previously by others but in more restricted contexts [7, 8]) presented the counterterms on spatial boundaries at early and late times that yield a finite action for asymptotically dS spacetimes in $3, 4, 5$ dimensions. By carrying out a procedure analogous to that in the AdS case [4, 5], one could get the boundary stress tensor on the spacetime boundary, and consequently a conserved charge interpreted as the mass of the asymptotically dS spacetime could be calculated. Sample calculations led the authors of [2] to the following conjecture: *Any asymptotically dS spacetime with mass greater than dS has a cosmological singularity.* Although an exact proof of this conjecture has not been attained, it has been verified for topological dS solutions and its dilatonic variants [9].

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The purpose of this paper is to investigate this prescription in greater detail. We first demonstrate in sections 2 and 3 that the procedure for deriving boundary counterterms from the Gauss-Codacci equation for asymptotically AdS spacetimes [10] applies also to the asymptotically dS case. We show that these counterterms are sufficient for obtaining a finite action for the inflationary patches (big bang and big crunch patches) of dS spacetime in any dimensionality. However such actions are *not* finite when computing for the full dS spacetime using covering coordinates: they contain a linear divergence in spacetimes of odd dimensionality. This divergence is similar to that found in the AdS case [11]. We then move on in section 4 to compute the action for a Schwarzschild-de Sitter (SdS) black hole with dimensionality up to nine. We also compute the boundary stress tensor and mass of these SdS black holes. We then define a notion of entropy outside of the horizon by generalizing the gravitational Gibbs-Duhem relation to this situation. By appropriately identifying a spatial coordinate outside of the horizon, infinite volume divergences due to integration over this coordinate on the boundary are removed, and our definition of entropy agrees with that obtained using CFT methods in 3 dimensions [2]. However the justification and interpretation of these results and the above conjecture is less than clear. For example masses greater than that of pure de Sitter spacetime can be obtained by reversing the sign of the mass parameter, whilst keeping all singularities hidden from observers outside of the cosmological horizon. We comment on this in the final section.

2. Boundary counterterms

In $d+1$ dimensions, the Einstein equations of motion with a positive cosmological constant can be derived from the action

$$S = I_B + I_{\partial B}\,, \tag{2.1}$$

where

$$I_B = \frac{\alpha}{16\pi G}\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\,(R - 2\Lambda + \mathcal{L}_M) \tag{2.2}$$

$$I_{\partial B} = \frac{\beta}{16\pi G}\int_{\partial\mathcal{M}^-}^{\partial\mathcal{M}^+} d^dx\sqrt{h^{\pm}}K^{\pm} \tag{2.3}$$

and $\mathcal{L}_M$ refers to the matter lagrangian, which we shall not consider here.

The first term in (2.1) is the bulk action over the $d+1$ dimensional Manifold $\mathcal{M}$ with newtonian constant G and the second term (2.3) is the Gibbons-Hawking surface term which is a necessary term to ensure a well defined Euler-Lagrange variation. $\partial\mathcal{M}^{\pm}$ are spatial euclidean boundaries at early and late times and $\int_{\partial\mathcal{M}^-}^{\partial\mathcal{M}^+} d^dx$ indicates an integral over the late time boundary minus an integral over the early time boundary. The quantities $g_{\mu\nu}, h^{\pm}_{\mu\nu}$ and $K^{\pm}$ are the bulk spacetime metric, induced boundary metrics and the trace of extrinsic curvatures of the boundaries respectively. We shall usually suppress the "$\pm$" notation when it is obvious. For a well-defined variational principle, we must have $\beta = -2\alpha$, and one typically chooses $\alpha = 1$ as an overall normalization.[1] However, as is well known the action (2.1) is not finite when evaluated on a solution of the equations of motion. The reason is the infinite volume of the spacetime at early and late times.

[1]Our conventions are the same as in ref. [10].

The procedure for dealing with such divergences in asymptotically flat/AdS cases (where they are large-distance effects) was to include a reference action term [3, 4], which corresponded to the action of embedding the boundary hypersurface $\partial\mathcal{M}$ (whose unit normal is spacelike) into some other manifold. The physical interpretation is that one has a collection of observers located on the closed manifold $\partial\mathcal{M}$, and that the physical quantities they measure (energy, angular momentum, etc.) are those contained within this closed manifold relative to those of some reference spacetime (regarded as the ground state) in which $\partial\mathcal{M}$ is embedded [12]. For example in an asymptotically anti de Sitter spacetime, it would be natural to take pure AdS as the ground state reference manifold.

However this procedure suffers from several drawbacks: the reference spacetime in general cannot be uniquely chosen [13] nor can an arbitrary boundary $\partial\mathcal{M}$ always be embedded in a given reference spacetime. Employing approximate embeddings can lead to ambiguity, confusion and incompleteness; examples of this include the Kerr [14], Taub-NUT and Taub-bolt spacetimes [15].

An alternative approach for asymptotically AdS spacetimes was suggested a few years ago that has enjoyed a greater measure of success [5, 16, 17, 18]. It involves adding to the action terms that depend only on curvature invariants that are functionals of the intrinsic boundary geometry. Such terms cannot alter the equations of motion and, since they are divergent, offer the possibility of removing divergences that arise in the action (2.1) provided the coefficients of the allowed curvature invariants are correctly chosen. No embedding spacetime is required, and computations of the action and conserved charges yield unambiguous finite values that are intrinsic to the spacetime. This has been explicitly verified for the full range of type-D asymptotically AdS spacetimes, including Schwarzchild-AdS, Kerr-AdS, Taub-NUT-AdS, Taub-bolt-AdS, and Taub-bolt-Kerr-AdS [11, 19, 20].

The boundary counterterm action is universal, and a straightforward algorithm has been constructed for generating it [10]. The procedure involves rewriting the Einstein equations in Gauss-Codacci form, and then solving them in terms of the extrinsic curvature functional $\Pi_{ab} = K_{ab} - Kh_{ab}$ of the boundary $\partial\mathcal{M}$ and its normal derivatives to obtain the divergent parts. It succeeds because all divergent parts can be expressed in terms of intrinsic boundary data, and do not depend on normal derivatives [21]. By writing the divergent part $\tilde{\Pi}_{ab}$ as a power series in the inverse cosmological constant the entire divergent structure can be covariantly isolated for any given boundary dimension d; by varying the boundary metric under a Weyl transformation, it is straightforward to show that the trace $\tilde{\Pi}$ is proportional to the divergent boundary counterterm lagrangian.

Explicit calculations have demonstrated that finite values for the action and conserved charges can be unambiguously computed up to $d = 8$ for the class of Kerr-AdS metrics [22]. The removal of divergences is completely analogous to that which takes place in quantum field theory by adding counterterms which are finite polynomials in the fields. The AdS/CFT correspondence conjecture asserts that these procedures are one and the same. Corroborative evidence for this is given by calculations which illustrate that the trace anomalies and Casimir energies obtained from the two different descriptions are in agreement for known cases [16, 17, 23].

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Generalizations of the counterterm action to asymptotically flat spacetimes have also been proposed [19, 24]. They are quite robust, and allow for a full calculation of quasilocal conserved quantities in the Kerr solution [25] that go well beyond the slow-rotating limit that approximate embedding techniques require [14]. Although they can be inferred for general d by considering spacetimes of special symmetry, they cannot be algorithmically generated, and are in general dependent upon the boundary topology [10].

Turning next to the asymptotically de Sitter case, we must add to the action (2.1) some counterterms to cancel its divergences

$$I_{ct} = \frac{1}{16\pi G}\int_{\partial\mathcal{M}^+} d^dx\sqrt{h}\mathcal{L}_{ct} + \frac{1}{16\pi G}\int_{\partial\mathcal{M}^-} d^dx\sqrt{h}\mathcal{L}_{ct} \tag{2.4}$$

so that

$$I = I_B + I_{\partial B} + I_{ct} \tag{2.5}$$

is now the total action.

For the special cases $d = 2, 3, 4$, the counterterm lagrangian

$$\mathcal{L}_{ct} = \gamma\left(-\frac{d-1}{\ell} + \frac{\ell\Theta\,(d-3)}{2(d-2)}\hat{R}\right) \tag{2.6}$$

was proposed [2], where $\hat{R}$ is the intrinsic curvature of the boundary surfaces and the step function $\Theta(x)$ is equal to zero unless $x \geq 0$ which in this case it equals unity. The parameter γ must equal -2α to cancel divergences. The action (2.6) was shown to cancel divergences in de Sitter spacetime

$$ds^2 = -d\tau^2 + \ell^2 \exp\left(\tau^2/\ell^2\right) d\vec{x}\cdot d\vec{x} \tag{2.7}$$

and the Nariai spacetime

$$ds^2 = -\left(\frac{d\tau^2}{\ell^2} - 1\right)^{-1} d\tau^2 + \left(\frac{d\tau^2}{\ell^2} - 1\right) dt^2 + \ell^2\left(1 - \frac{2}{d}\right) d\Omega_{d-1}^2\,, \tag{2.8}$$

where the metric $d\vec{x}\cdot d\vec{x}$ is a flat d-dimensional metric that covers an inflationary patch of de Sitter spacetime and $d\Omega_{d-1}^2$ is the metric of a unit $(d-1)$-sphere. Here

$$\Lambda = \frac{d\,(d-1)}{2\ell^2} \tag{2.9}$$

is the positive cosmological constant.

These results are suggestive that an algorithm similar to that in the AdS case is applicable here, and indeed this is the case. Following the procedure in ref. [10], we write the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu} \tag{2.10}$$

in the Gauss-Codacci form

$$\widehat{R}_{ab} - \frac{1}{2}\widehat{R}h_{ab} + u^\mu\nabla_\mu\Pi_{ab} - \frac{1}{2}h_{ab}\left(\frac{\Pi^2}{d-1} - \Pi_{cd}\Pi^{cd}\right) + \frac{\Pi}{d-1}\Pi_{ab} = \frac{d\,(d-1)}{2\ell^2}h_{ab} \tag{2.11}$$

$$\nabla^b\Pi_{ab} = 0 \tag{2.12}$$

$$\frac{1}{2}\left(\frac{\Pi^2}{d-1} - \Pi_{cd}\Pi^{cd} - \hat{R}\right) = \frac{d\,(d-1)}{2\ell^2}\,, \tag{2.13}$$

where $u^\mu_\pm$ is the timelike unit normal to $\partial\mathcal{M}^\pm$, whose metric is $h^\pm_{ab}$; eqs.(2.11)–(2.13) are valid for each of these submanifolds. From the work of Mottola and Mazur [26], we know that the divergences of asymptotically de Sitter spacetimes are independent of the boundary normal, and so depend only on intrinsic boundary data. By writing the divergent part $\tilde{\Pi}_{ab}$ as a power series in ℓ

$$\tilde{\Pi}_{ab} = \sum_{n=0}^{[d/2]} \ell^{2n-1}\tilde{\Pi}^{(n)}_{ab} \tag{2.14}$$

it is easy to show that the trace $\tilde{\Pi}^{a(n)}_a$ appears linearly in eq. (2.13), and so can be determined inductively in terms all $\tilde{\Pi}^{(k)}_{ab}$, $k<n$, if these are known. However these can be determined from the counterterm lagrangian provided

$$\tilde{\Pi}_{ab} = \frac{2}{\sqrt{h}}\frac{\delta}{\delta h_{ab}}\int d^dx\sqrt{h}\mathcal{L}_{ct} \tag{2.15}$$

so that under a Weyl rescaling $\delta_W h_{ab} = \sigma h_{ab}$ we obtain after some algebra

$$(d-2n)\,\mathcal{L}^{(n)}_{ct} = \tilde{\Pi}^{a(n)}_a \tag{2.16}$$

up to an irrelevant total divergence, where $\mathcal{L}_{ct} = \sum_{n=0}^{[d/2]}\ell^{2n-1}\mathcal{L}^{(n)}_{ct}$.

The procedure for finding the counterterm lagrangian for any given d is identical to the AdS case. Setting

$$\tilde{\Pi}^{(0)}_{ab} = (1-d)h_{ab} \tag{2.17}$$

we obtain

$$\mathcal{L}^{(0)}_{ct} = (1-d) \tag{2.18}$$

from (2.16). Using this we insert the series (2.14) into eq. (2.13) and inductively obtain

$$\begin{aligned}\mathcal{L}_{ct} = &\left(-\frac{d-1}{\ell}+\frac{\ell\Theta\,(d-3)}{2(d-2)}\hat{R}\right) - \frac{\ell^3\Theta\,(d-4)}{2(d-2)^2(d-4)}\left(\hat{R}^{ab}\hat{R}_{ab}-\frac{d}{4(d-1)}\hat{R}^2\right) - \\ &-\frac{\ell^5\Theta\,(d-5)}{(d-2)^3(d-4)(d-6)}\left(\frac{3d+2}{4(d-1)}\hat{R}\hat{R}^{ab}\hat{R}_{ab}-\frac{d(d+2)}{16(d-1)^2}\hat{R}^3-2\hat{R}^{ab}\hat{R}^{cd}\hat{R}_{acbd}-\right.\\ &\left.-\frac{d}{4(d-1)}\nabla_a\hat{R}\nabla^a\hat{R}+\nabla^c\hat{R}^{ab}\nabla_c\hat{R}_{ab}\right)\end{aligned} \tag{2.19}$$

for $d\le 8$. The associated boundary stress-energy tensor can be obtained by the variation of the action with respect to the variation of the boundary metric, and is given by:

$$\begin{aligned}-8\pi GT_{ab} = &\left(K_{ab}-Kh_{ab}\right)+\Bigg\{\left(\frac{d-1}{\ell}h_{ab}+\frac{\ell\Theta\,(d-3)}{(d-2)}G_{ab}\right)-\frac{\ell^3\Theta\,(d-4)}{(d-2)^2(d-4)}\times\\ &\times\left[-\frac{1}{2}h_{ab}\left(\hat{R}^{cd}\hat{R}_{cd}-\frac{d}{4(d-1)}\hat{R}^2\right)-\frac{d}{2(d-1)}\hat{R}\hat{R}_{ab}-\right.\\ &\left.-\frac{1}{2(d-1)}h_{ab}\nabla^2\hat{R}+2\hat{R}^{cd}\hat{R}_{cadb}-\frac{d-2}{2(d-1)}\nabla_a\nabla_b\hat{R}+\nabla^2\hat{R}_{ab}\right]-\end{aligned}$$

$$
\begin{aligned}
&-\frac{2\ell^5\Theta\,(d-5)}{(d-2)^3(d-4)(d-6)}\times \\
&\times\left\{\frac{3d+2}{4(d-1)}\left(G_{ab}\hat{R}^{cd}\hat{R}_{cd}-\nabla_a\nabla_b\left(\hat{R}^{ef}\hat{R}_{ef}\right)+h_{ab}\nabla^2\left(\hat{R}^{ef}\hat{R}_{ef}\right)\right)+\right.\\
&\quad+2\hat{R}\hat{R}_a{}^c\hat{R}_{bc}+h_{ab}\nabla_c\nabla_d\left(\hat{R}\hat{R}^{cd}\right)+\nabla^2\left(\hat{R}\hat{R}_{ab}\right)-\nabla^c\nabla_b\left(\hat{R}\hat{R}_{ac}\right)-\\
&\quad-\nabla^c\nabla_a\left(\hat{R}\hat{R}_{bc}\right)-\frac{d(d+2)}{16(d-1)^2}\times\\
&\quad\times\left[-\frac{1}{2}h_{ab}\hat{R}^3+3\hat{R}^2\hat{R}_{ab}-3\nabla_a\nabla_b\hat{R}^2+3h_{ab}\nabla^2\hat{R}^2\right]-\\
&\quad-2\left[-\frac{1}{2}h_{ab}\hat{R}^{ef}\hat{R}^{cd}\hat{R}_{ecfd}+\frac{3}{2}-\left(-\hat{R}_a{}^e\hat{R}^{cd}\hat{R}_{ecbd}+\hat{R}_b{}^e\hat{R}-^{cd}\hat{R}_{ecad}\right)-\right.\\
&\qquad-\nabla_c\nabla_d\left(-\hat{R}_{ab}\hat{R}^{cd}\right)+\nabla_c\nabla_d\left(-\hat{R}_a{}^c\hat{R}_b{}^d\right)+\\
&\qquad+h_{ab}\nabla_e\nabla^f\left(\hat{R}^{cd}\hat{R}^e{}_{cfd}\right)+\nabla^2\left(\hat{R}^{cd}\hat{R}_{acbd}\right)-\nabla_e\nabla_a\left(\hat{R}^{cd}\hat{R}^e{}_{cbd}\right)-\\
&\qquad\left.-\nabla_e\nabla_b\left(\hat{R}^{cd}\hat{R}^e{}_{cad}\right)\right]-\frac{d}{4(d-1)}\times\\
&\quad\times\left[\nabla_a\hat{R}\nabla_b\hat{R}-\frac{1}{2}h_{ab}\left(\nabla_c\hat{R}\nabla^c\hat{R}\right)-2\hat{R}_{ab}\nabla^2\hat{R}-2h_{ab}\nabla^4\hat{R}+\right.\\
&\qquad\left.+2\nabla_a\nabla_b\nabla^2\hat{R}\right]+2\nabla_c\hat{R}_{ad}\nabla^c\hat{R}_b{}^d+\nabla_a\hat{R}^{cd}\nabla_b\hat{R}_{cd}-\\
&\quad-\frac{1}{2}h_{ab}\nabla^e\hat{R}^{cd}\nabla_e\hat{R}_{cd}-h_{ab}\nabla_c\nabla_d\nabla^2\hat{R}^{cd}-\nabla^4\hat{R}_{ab}+\nabla_c\nabla_a\nabla^2\hat{R}_b{}^c+\\
&\quad+\nabla_c\nabla_b\nabla^2\hat{R}_a{}^c-\nabla_c\left(\hat{R}_{bd}\nabla_a\hat{R}^{cd}\right)-\nabla_c\left(\hat{R}_{ad}\nabla_b\hat{R}^{cd}\right)- \qquad (2.20)\\
&\quad\left.-\nabla_c\left(\hat{R}_{ad}\nabla^c\hat{R}_b{}^d+\hat{R}_{bd}\nabla^c\hat{R}_a{}^d\right)+\nabla^c\left(\hat{R}_{cd}\nabla_a\hat{R}_b{}^d\right)+\nabla^c\left(\hat{R}_{cd}\nabla_b\hat{R}_a{}^d\right)\right\}\Bigg\}.
\end{aligned}
$$

If the boundary geometry has an isometry generated by a Killing vector ξ^μ, then it is straightforward to show that $T_{ab}\xi^b$ is divergenceless. We write the boundary metric in the form

$$
h_{ab}d\hat{x}^a d\hat{x}^b = d\hat{s}^2 = N_t^2 dt^2 + \sigma_{ab}\left(d\varphi^a + N^a dt\right)\left(d\varphi^b + N^b dt\right), \qquad (2.21)
$$

where $\nabla_\mu t$ is a spacelike vector field that is the analytic continuation of a timelike vector field and the φ^a are coordinates describing closed surfaces Σ. From this it is straightforward to show that the quantity

$$
\mathfrak{Q} = \oint_\Sigma d^{d-1}\varphi\sqrt{\sigma}n^a T_{ab}\xi^b \qquad (2.22)
$$

is conserved between surfaces of constant t, whose unit normal is given by n^a. Physically this would mean that a collection of observers on the hypersurface whose metric is h_{ab} would all observe the same value of $\mathfrak{Q}$ provided this surface had an isometry generated by ξ^b. If $\partial/\partial t$ is itself a Killing vector, then we can define

$$\mathfrak{M} = \oint_{\Sigma} d^{d-1}\varphi\sqrt{\sigma}N_t n^a n^b T_{ab} \tag{2.23}$$

as the conserved mass associated with the surface Σ at any given point t on the boundary. This quantity changes with the cosmological time τ. However a collection of observers that defined a surface Σ would find that the value of $\mathfrak{M}$ that they would measure would not change as they collectively relocated to a different value of t on the spacelike surface $\partial\mathcal{M}$. Since all asymptotically de Sitter spacetimes must have an asymptotic isometry generated by $\partial/\partial t$, there is at least the notion of a conserved total mass $\mathfrak{M}$ for the spacetime as computed at future/past infinity. Similarly the quantity

$$\mathfrak{J}^a = \oint_{\Sigma} d^{d-1}\varphi\sqrt{\sigma}\sigma^{ab}n^c T_{bc} \tag{2.24}$$

can be regarded as a conserved angular momentum associated with the surface Σ if the surface has an isometry generated by $\partial/\partial\phi^a$.

3. Actions in de Sitter spacetime

We consider in this section an evaluation of the action using the prescription (2.19). From equation (2.10), one gets

$$R = -\Lambda\frac{d+1}{(1-(d+1)/2)} = -\frac{d(d-1)(d+1)}{(1-d)\ell^2} = \frac{d(d+1)}{\ell^2} \tag{3.1}$$

so that

$$I_B = \frac{\alpha}{16\pi G}\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\left(\frac{d(d+1)}{\ell^2} - \frac{d(d-1)}{\ell^2}\right) = \frac{d\alpha}{8\pi G\ell^2}\int_{\mathcal{M}_d} d^d\hat{x}\int d\tau\sqrt{f}\sqrt{h}\,, \tag{3.2}$$

where $V_d = \int_{\mathcal{M}_d} d^d\widehat{x}$ is the volume of the d dimensional spatial section, and τ is the orthogonal coordinate direction.

The metric is of the form

$$ds^2 = -f(\tau)\,d\tau^2 + d\hat{s}^2\,, \tag{3.3}$$

where $d\hat{s}^2$ is given by (2.21). Hence the timelike vector normal to the hypersurface is

$$u^\mu = \left(f^{-1/2}(\tau), 0, 0, \ldots, 0\right) \tag{3.4}$$

which yields

$$\begin{aligned} K &= h^{\mu\nu}\nabla_\mu u_\nu = -h^{\mu\nu}\Gamma^\lambda_{\mu\nu}u_\lambda = -\frac{1}{2\sqrt{f}}h^{\mu\nu}\left(\partial_\nu h_{\mu\tau} + \partial_\mu h_{\nu\tau} - \partial_\tau h_{\mu\nu}\right) \\ &= \frac{1}{2\sqrt{f}}h^{\mu\nu}\partial_\tau h_{\mu\nu} \end{aligned} \tag{3.5}$$

in turn giving

$$I_{\partial B} = \frac{\beta}{32\pi G\sqrt{f}} \int_{\mathcal{M}_d^-}^{\mathcal{M}_d^+} d^d\widehat{x}\sqrt{h}\,(h^{\mu\nu}\partial_\tau h_{\mu\nu})\,. \tag{3.6}$$

So we finally get

$$\begin{aligned} I &= I_B + I_{\partial B} + I_{ct} \\ &= \frac{1}{16\pi G\ell^2}\int_{\mathcal{M}_d^-}^{\mathcal{M}_d^+} d^d\hat{x}\left[\left(2d\alpha\int d\tau\sqrt{h}\sqrt{f}\right) + \sqrt{h}\left(\frac{\beta}{2\sqrt{f}}\,(h^{\mu\nu}\partial_\tau h_{\mu\nu}) + \mathcal{L}_{ct}\right)\right] \end{aligned} \tag{3.7}$$

for the generic form of the action. We turn next to its evaluation in de Sitter spacetime.

3.1 Inflationary coordinates

The dS spacetime admits a coordinate system where equal time surfaces are flat. In this case

$$ds^2 = -d\tau^2 + e^{2\tau/\ell}d\vec{x}^2 \tag{3.8}$$

is the solution to the Einstein equations in de Sitter coordinates. τ changes from $-\infty$ to $+\infty$, and this patch (called the big bang patch) covers half of the dS spacetime from a big bang at a past horizon to the euclidean surface at future infinity. The other half (big crunch patch) of the dS spacetime from past infinity to a future horizon could be obtained by replacing τ by $-\tau$ in (3.8). So, comparing with (3.3), $h^+_{\mu\nu}$ is a flat metric, and the counterterm lagrangian reduces to its first term for any d. Hence we have

$$\begin{aligned} I &= \frac{1}{16\pi G}\int_{\mathcal{M}_d^+} d^d\hat{x}\left[2d\alpha\int_{-\infty}^{+\infty}\frac{d\tau}{\ell^2}e^{d\tau/\ell} + \frac{\beta}{2}\,(2d/\ell)\;e^{d\tau/\ell}\Big|_{-\infty}^{+\infty} + \gamma\;e^{d\tau/\ell}\Big|_{\tau=+\infty}\left(\frac{1-d}{\ell}\right)\right] \\ &= \frac{V_d}{16\pi G\ell}\left[2\alpha + \frac{\beta}{2}\,(2d) + \gamma\,(1-d)\right]\left(e^{d\tau/\ell}\Big|_{-\infty}^{\infty}\right), \end{aligned} \tag{3.9}$$

where $V_d = \int_{\mathcal{M}_d^+} d^d\hat{x}$. This will diverge at $\tau = +\infty$ unless

$$\beta = \gamma = -2\alpha \tag{3.10}$$

in which case it vanishes. The quantity V_d will also diverge unless the boundary rendered compact, e.g. by toroidal identifications.

So, for every dS spacetime in big bang coordinates the counterterm lagrangian (2.19) removes all the divergences of the action (2.5) in any dimension. A similar calculation shows that in the big crunch coordinates with choosing (3.10) the divergences of the action are removed.

3.2 Covering coordinates

Now we use global coordinates of the dS spacetime for which equal time hypersurfaces are d-spheres S^d

$$ds^2 = -d\tau^2 + \ell^2\cosh^2(\tau/\ell)\,d\hat{\Omega}_d^2\,. \tag{3.11}$$

These hypersurfaces have an infinitely large radius at $\tau = -\infty$, which decrease to a minimum value ℓ as $\tau \to 0$, increasing again to infinity for $\tau = +\infty$. Here $h_{\mu\nu}$ is the metric of the d-sphere, so we have

$$\begin{aligned} \hat{R}_{acbd} &= (h_{ab}h_{cd} - h_{ad}h_{bc}) \qquad \hat{R}_{ab} = (d-1)\,h_{ab} \qquad \hat{R} = d\,(d-1) \\ \hat{R}_{ab}\hat{R}^{ab} &= (d-1)^2\,d \qquad \hat{R}^{ab}\hat{R}^{cd}\hat{R}_{acbd} = d\,(d-1)^3\ . \end{aligned} \tag{3.12}$$

This in turn yields the following counterterm lagrangian (2.19)

$$\begin{aligned} \mathcal{L}_{ct} = \gamma\Bigg[&\left(-\frac{d-1}{\ell} + \frac{\Theta\,(d-3)\,d\,(d-1)}{2\ell(d-2)\cosh^2(\tau/\ell)}\right) + \frac{\Theta\,(d-4)\,d(d-1)}{8\ell(d-4)\cosh^4(\tau/\ell)} + \\ &+ \frac{\Theta\,(d-5)\,(d-1)\,d}{16\ell(d-6)\cosh^6(\tau/\ell)}\Bigg] \end{aligned} \tag{3.13}$$

and so the total action is

$$\begin{aligned} I = &\frac{\ell^{d-1}V_d}{16\pi G}\times \\ &\times\Bigg[2d\alpha\int_{-T}^{T} d\,(\tau/\ell)\cosh^d(\tau/\ell) + 2\frac{\beta}{2}\,(2d)\cosh^{d-1}(T/\ell)\sinh(T/\ell) + \\ &+ 2\gamma\Bigg\{\cosh^d(T/\ell)\left(1-d+\frac{\Theta\,(d-3)\,d\,(d-1)}{2(d-2)\cosh^2(T/\ell)}\right) + \\ &+ \frac{\Theta\,(d-4)\,d(d-1)\cosh^{d-4}(T/\ell)}{8(d-4)} + \frac{\Theta\,(d-5)\,(d-1)\,d\cosh^{d-6}(T/\ell)}{16(d-6)}\Bigg\}\Bigg] \end{aligned} \tag{3.14}$$

up to $d = 8$. Using (3.10), and setting $\alpha = 1$, we obtain

$$\begin{aligned} I^{d=2} &= \frac{(4T/\ell + 2)}{16\pi G}\ell V_2 \\ I^{d=3} &= 0 \\ I^{d=4} &= \frac{(6T/\ell - 3/2)}{16\pi G}\ell^3 V_4 \\ I^{d=5} &= 0 \\ I^{d=6} &= \frac{(15T/2\ell - 25/8)}{16\pi G}\ell^5 V_6 \\ I^{d=7} &= 0 \\ I^{d=8} &= \frac{(35T/4\ell - 413/96)}{16\pi G}\ell^7 V_8 \end{aligned} \tag{3.15}$$

for the action (3.14) in the different dimensionalities.

We see that the action is finite up to a term that diverges linearly with T for even d as $T \to \infty$. This divergence cannot be removed by a judicious choice of counterterms that are polynomials in boundary curvature invariants because such invariants are all independent of τ, as illustrated in eq. (3.12). Clearly there are limitations to the counterterm prescription. We shall comment on the implications of this in terms of a possible dS/CFT correspondence in the concluding section 5.

4. Schwarzchild-dS spacetimes

In this section, we consider the $d+1$ dimensional SdS spacetime. The metric is

$$ds^2 = -N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\hat{\Omega}^2_{d-1}\,, \tag{4.1}$$

where

$$N(r) = 1 - \frac{2m}{r^{d-2}} - \frac{r^2}{\ell^2} \tag{4.2}$$

and $d\hat{\Omega}^2_{d-1}$ denotes the metric on the unit sphere S^{d-1}. For mass parameters m with $0 < m < m_N$, where

$$m_N = \frac{\ell^{d-2}}{d}\left(\frac{d-2}{d}\right)^{\frac{d-2}{2}} \tag{4.3}$$

we have a black hole in dS spacetime with event horizon at $r = r_H$ and cosmological horizon at $r = r_C > r_H$. The event and cosmological horizons locate in $N(r_H) = N(r_C) = 0$. When $m = m_N$, the event horizon coincides with the cosmological horizon and one gets the Nariai solution. For $m > m_N$, the metric (4.1) describes a naked singularity in an asymptotically dS spacetime. So demanding the absence of naked singularities yields an upper limit to the mass of the SdS black hole. We want to work outside of the cosmological horizon, where $N(r) < 0$, so we set $r = \tau$ and rewrite the metric as

$$ds^2 = -f(\tau)d\tau^2 + \frac{dt^2}{f(\tau)} + \tau^2 d\tilde{\Omega}^2_{d-1}\,, \tag{4.4}$$

where

$$f(\tau) = \left(\frac{\tau^2}{\ell^2} + \frac{2m}{\tau^{d-2}} - 1\right)^{-1} . \tag{4.5}$$

The bulk action is now

$$\begin{aligned} I_B &= \frac{d}{8\pi G\ell^2}\int d^dx \int_{\tau_+}^{\tau} d\tau \sqrt{f}\sqrt{h} = \frac{d}{8\pi G\ell^2}\int dt d^{d-1}\hat{x}\sqrt{\sigma}\int_{\tau_+}^{\tau} d\tau \tau^{d-1} \\ &= \frac{V^t_{d-1}}{8\pi G\ell^2}\left(\tau^d - \tau_+^d\right)\,, \end{aligned} \tag{4.6}$$

where $V^t_{d-1} = \int dt d^{d-1}\hat{x}\sqrt{\sigma}$, where σ^{ab} is the metric on the unit $(d-1)$-sphere. Here τ_+ is the location of cosmological horizon which defined so that τ_+ is the largest root of $[f(\tau_+)]^{-1} = 0$; the integration is from the cosmological horizon out to some fixed τ that will be sent to infinity. We shall work in this "upper patch" outside of the cosmological horizon in SdS spacetime; results for the lower patch are obtained in a similar manner by setting $r = -\tau$ and considering $-\infty < \tau < -\tau_+$.

The trace of the extrinsic curvature is

$$\begin{aligned} K &= \frac{1}{2\sqrt{f}}h^{\mu\nu}\partial_\tau h_{\mu\nu} = \frac{\sqrt{f}}{2}\partial_\tau\left(\frac{1}{f}\right) + \frac{(d-1)\,\tau^{-2}}{2\sqrt{f}}\partial_\tau \tau^2 \\ &= \frac{1}{2\sqrt{f}}\left(-\frac{f'}{f} + \frac{2\,(d-1)}{\tau}\right) \end{aligned} \tag{4.7}$$

and so the boundary action becomes

$$\begin{aligned} I_{\partial B} &= -\frac{1}{16\pi G f}\int dt d^{d-1}\hat{x}\sqrt{\sigma}\tau^{d-1}\left(-\frac{f'}{f}+\frac{2\,(d-1)}{\tau}\right) \\ &= -\frac{V^t_{d-1}}{16\pi G f}\tau^{d-1}\left(-\frac{f'}{f}+\frac{2\,(d-1)}{\tau}\right), \end{aligned} \tag{4.8}$$

where (3.10) with $\alpha = 1$, has been employed.

Here $h_{\mu\nu}$ is the product metric of the $(d-1)$-sphere σ_{ab} with dt, so we have

$$\begin{aligned} \hat{R}_{acbd} &= (\sigma_{ab}\sigma_{cd} - \sigma_{ad}\sigma_{bc}) \qquad \hat{R}_{ab} = (d-2)\,\sigma_{ab} \qquad \hat{R} = (d-2)\,(d-1) \\ \hat{R}_{ab}\hat{R}^{ab} &= (d-2)^2\,(d-1) \qquad \hat{R}^{ab}\hat{R}^{cd}\hat{R}_{acbd} = (d-1)\,(d-2)^3 \\ \nabla_c\hat{R}_{ab} &= \nabla_c\hat{R} = 0\,, \end{aligned} \tag{4.9}$$

where all t-components in any quantity in (4.9) vanish. Consequently

$$\begin{aligned} \mathcal{L}_{ct} &= \left(-\frac{d-1}{\ell}+\frac{\ell\Theta\,(d-3)}{2\tau^2}\,(d-1)\right) - \frac{\ell^3\Theta\,(d-4)}{2(d-4)\tau^4}\,(d-1)\left(1-\frac{d}{4}\right) - \\ &\quad -\frac{\ell^5\Theta\,(d-5)}{(d-4)(d-6)\tau^6}\,(d-1)\left[\frac{3d+2}{4}-\frac{d(d+2)}{16}-2\right] \\ &= \left(\frac{1-d}{\ell}+\frac{\ell\Theta\,(d-3)}{2\tau^2}\,(d-1)\right)+\frac{\ell^3\Theta\,(d-4)}{8\tau^4}\,(d-1)+\frac{\ell^5\Theta\,(d-5)}{16\tau^6}\,(d-1)\,. \end{aligned} \tag{4.10}$$

So using (3.10) the action becomes

$$\begin{aligned} I &= \frac{V^t_{d-1}}{8\pi G\ell^2}\left(\tau^d-\tau^d_+\right) - \frac{V^t_{d-1}}{16\pi G f}\tau^{d-1}\left(-\frac{f'}{f}+\frac{2\,(d-1)}{\tau}\right) - \\ &\quad -\frac{(d-1)\,V^t_{d-1}\tau^{d-1}}{8\pi G\sqrt{f}}\left[\left(-\frac{1}{\ell}+\frac{\ell\Theta\,(d-3)}{2\tau^2}\right)+\frac{\ell^3\Theta\,(d-4)}{8\tau^4}+\frac{\ell^5\Theta\,(d-5)}{16\tau^6}\right] \\ &= \frac{V^t_{d-1}\tau^{d-1}}{8\pi G}\left[\frac{1}{\ell^2}(\tau-\frac{\tau^d_+}{\tau^{d-1}})-\frac{1}{2f}\left(-\frac{f'}{f}+\frac{2\,(d-1)}{\tau}\right)-\frac{(d-1)}{\sqrt{f}}\times\right. \\ &\qquad \left.\times\left[\left(-\frac{1}{\ell}+\frac{\ell\Theta\,(d-3)}{2\tau^2}\right)+\frac{\ell^3\Theta\,(d-4)}{8\tau^4}+\frac{\ell^5\Theta\,(d-5)}{16\tau^6}\right]\right] \end{aligned} \tag{4.11}$$

and we obtain the following form of the actions in different dimensions

$$\begin{aligned} I^{d=2} &= -\frac{\left(m-1/2+\tau_+^2/2\ell^2\right)V^t_1}{4\pi G} \\ I^{d=3} &= -\frac{\left(m+\tau_+^3/2\ell^2\right)V^t_2}{4\pi G} \\ I^{d=4} &= -\frac{\left(m-3\ell^2/8+\tau_+^4/2\ell^2\right)V^t_3}{4\pi G} \\ I^{d=5} &= -\frac{\left(m+\tau_+^5/2\ell^2\right)V^t_4}{4\pi G} \\ I^{d=6} &= -\frac{\left(m-5\ell^4/16+\tau_+^6/2\ell^2\right)V^t_5}{4\pi G} \end{aligned}$$

$$
\begin{aligned}
I^{d=7} &= -\frac{\left(m+\tau_+^7/2\ell^2\right)V_6^t}{4\pi G} \\
I^{d=8} &= -\frac{\left(m-35\ell^6/128+\tau_+^8/2\ell^2\right)V_7^t}{4\pi G}
\end{aligned}
\tag{4.12}
$$

in the limit $\tau \to +\infty$. We note that all actions are finite. Note that the τ_+-independent terms for even d are consistently positive.

From (2.23) the mass is

$$
\mathfrak{M} = \int d^{d-1}\hat{x}\sqrt{\sigma}\tau^{d-1}N_t n^a n^b T_{ab} = \sqrt{f}\int d^{d-1}\hat{x}\sqrt{\sigma}\tau^{d-1}T_{tt}\,,
\tag{4.13}
$$

where

$$
n^a = (0, \sqrt{f}, \vec{0})
\tag{4.14}
$$

is the unit normal in the t-direction and $N_t = \frac{1}{\sqrt{f}}$. The extrinsic curvature $K_{ab} = h_a^\mu \nabla_\mu u_b$ is

$$
\begin{aligned}
K_{tt} &= h_t^\mu \nabla_\mu u_t = h_t^\mu\left(\partial_\mu u_t - \Gamma_{\mu t}^\lambda u_\lambda\right) = -\frac{1}{2\sqrt{f}}\left(2\partial_t h_{t\tau} - \partial_\tau h_{tt}\right) \\
&= \frac{1}{2\sqrt{f}}\partial_\tau h_{tt} = -\frac{f'}{2f^2\sqrt{f}}\,,
\end{aligned}
\tag{4.15}
$$

where the prime refers to the derivative with respect to τ.

Since there is constant curvature in the $(d-2)$-dimensional subspace and all t-components vanish from the curvatures in (4.9), we have

$$
\begin{aligned}
T_{tt} = \frac{1}{4\pi G}\Bigg[&(K_{tt} - Kh_{tt}) + \left(\frac{d-1}{\ell}h_{tt} - \frac{\ell\Theta\,(d-3)}{2(d-2)}h_{tt}\hat{R}\right) - \\
&- \frac{\ell^3\Theta\,(d-4)}{(d-2)^2(d-4)}\left\{\frac{1}{2}h_{tt}\left(\hat{R}^{cd}\hat{R}_{cd} - \frac{d}{4(d-1)}\hat{R}^2\right)\right\} - \frac{2\ell^5\Theta\,(d-5)}{(d-2)^3(d-4)(d-6)}\times \\
&\times\Bigg\{\frac{3d+2}{4(d-1)}\left(-\frac{1}{2}h_{tt}\hat{R}\hat{R}^{cd}\hat{R}_{cd}\right) - \frac{d(d+2)}{16(d-1)^2}\left[-\frac{1}{2}h_{tt}\hat{R}^3\right] - 2 - \\
&- \frac{1}{2}h_{tt}\hat{R}^{ef}\hat{R}^{cd}\hat{R}_{ecfd} + +h_{tt}\nabla_e\nabla^f\left(\hat{R}^{cd}\hat{R}^e{}_{cfd}\right) + \nabla^2\left(\hat{R}^{cd}\hat{R}_{tctd}\right) - \\
&- 2\nabla_e\nabla_t\left(\hat{R}^{cd}\hat{R}^e{}_{ctd}\right)\Bigg\}\Bigg]
\end{aligned}
\tag{4.16}
$$

or

$$
\begin{aligned}
T_{tt} = \frac{1}{4\pi G}\Bigg[&-\frac{1}{2f\sqrt{f}}\left(\frac{2\,(d-1)}{\tau}\right) + \left(\frac{d-1}{f\ell} - \frac{\ell\Theta\,(d-3)}{2f\tau^2}\,(d-1)\right) - \\
&- \frac{\ell^3\Theta\,(d-4)}{8f\tau^4}\,(d-1) - \frac{2\ell^5\Theta\,(d-5)}{32f\tau^6}(d-1)\Bigg]\,.
\end{aligned}
\tag{4.17}
$$

From this component of stress tensor, using (4.13), we obtain

$$
\mathfrak{M}^{d=2} = -\frac{V_1}{8\pi G}(2m-1)\left\{1 + \frac{1}{4\tau^2}\ell^2(2m-1) + \mathcal{O}\left(\frac{1}{\tau^4}\right)\right\}
$$

$$
\begin{aligned}
\mathfrak{M}^{d=3} &= -\frac{V_2}{8\pi G}\left(4m - \frac{\ell^2}{2\tau} + \mathcal{O}\left(\frac{1}{\tau^3}\right)\right)\\
\mathfrak{M}^{d=4} &= -\frac{V_3}{8\pi G}\left(6m - \frac{3}{4}\ell^2 + \frac{3\ell^2}{64\tau^4}(8m-\ell^2)^2 + \mathcal{O}\left(\frac{1}{\tau^6}\right)\right)\\
\mathfrak{M}^{d=5} &= -\frac{V_4}{8\pi G}\left(8m - \frac{\ell^4}{2\tau} + \mathcal{O}\left(\frac{1}{\tau^3}\right)\right)\\
\mathfrak{M}^{d=6} &= -\frac{V_5}{8\pi G}\left(10m - \frac{5}{8}\ell^4 - \frac{5\ell^6}{64\tau^2} + \mathcal{O}\left(\frac{1}{\tau^6}\right)\right)\\
\mathfrak{M}^{d=7} &= -\frac{V_6}{8\pi G}\left(12m - \frac{15}{32}\frac{\ell^6}{\tau} + \mathcal{O}\left(\frac{1}{\tau^3}\right)\right)\\
\mathfrak{M}^{d=8} &= -\frac{V_7}{8\pi G}\left(14m - \frac{35}{64}\ell^6 - \frac{7}{64}\frac{\ell^8}{\tau^2} + \mathcal{O}\left(\frac{1}{\tau^4}\right)\right),
\end{aligned}
\tag{4.18}
$$

where we have retained the leading terms in τ in the large-τ limit. For odd values of d, $\mathfrak{M}$ is an increasingly negative function of τ, approaching a constant negative value as $\tau \to \infty$. For even values of d the situation is reversed: $\mathfrak{M}$ is an increasingly positive function of τ, approaching a constant positive value as $\tau \to \infty$. As the mass parameter m increases, this constant positive value decreases, approaching its minimum at the Nariai limit. Setting $m = 0$ gives the mass of dS spacetime in different dimensionalities. We note that dS spacetime with even dimensions has zero mass and the others with even dimensions have positive mass. Our results in the special case of $d = 4$ agrees with the known Casimir energy of the dual CFT living on the boundary of dS_5 [7] (up to a sign, because our signs are the same as [2]). Also, we note that our mass formula in the special case of $d = 2$ agrees with the result of [8]. Furthermore, we observe that if the dual CFT theory exists [27, 28] (like the CFT dual to AdS spacetime), then the dual of the calculated mass $\mathfrak{M}^d$ (4.18) is the energy of a boundary euclidean CFT_d. Since our mass $\mathfrak{M}^d$ decreases with increasing black hole mass parameter m, so the entropy of the dual boundary theory which is proportional to the energy of the boundary CFT, decreases relative to its de Sitter maximum.

The volumes V^t_{d-1} are in general divergent, since the t-coordinate is of infinite range. However since $\partial/\partial t$ is a Killing vector, it is tempting to periodically identify it. Indeed, if we analytically continue $t \to it$, we obtain a metric of signature $(-2, d-1)$. The submanifold of signature $(-,-)$ described by the (t,τ) coordinates will have a conical singularity at $\tau = \tau_+$ unless the t-coordinate is periodically identified with period

$$
\beta_H = \left|\frac{4\pi}{(-N'(r))}\right|_{r=\tau_+} = \left|\frac{-4\pi f'(\tau)}{f^2}\right|_{\tau=\tau_+}. \tag{4.19}
$$

This is the analogue of the Hawking temperature outside of the cosmological horizon.

Proceeding further, we can provisionally define an 'entropy' by analytically continuing the gravitational Gibbs-Duhem relation [15]:

$$
S = \beta_H \mathfrak{M}_{\tau\to\infty} - I\,, \tag{4.20}
$$

where $\beta_H = \oint dt$ is the euclideanized integral over t.This gives

$$
S_d = \frac{\left(\tau_+^d - 2(d-2)m\ell^2\right)\beta_H V_{d-1}}{8\pi G\ell^2} \tag{4.21}
$$

up to $d = 8$. It is straightforward to show that these entropies are always positive, since $\tau_+^d > 2(d-2)m\ell^2$ so long as $m < m_N$.

For example, for $d = 2$, we have

$$\frac{\tau_+^2}{\ell^2} + 2m - 1 = 0 \Rightarrow \tau_+ = \ell\sqrt{1-2m}$$
$$\Rightarrow S^{d=2} = \frac{(1-2m)\,\beta_H V_1}{8\pi G} \tag{4.22}$$

and from (4.19) we have $\beta_H = 2\pi\ell^2/\tau_+$, so

$$S^{d=2} = \frac{\tau_+ V_1}{4G} = \frac{\pi\ell\sqrt{1-2m}}{2G} \tag{4.23}$$

in agreement with ref. [2] and [8], provided we set $1 - 2m \to M$, which is the metric for a conical deficit. Moreover, from expressions (4.21), we see that in any dimension up to nine, the entropy of SdS black hole is monotonically decreasing function of the mass parameter. Hence the entropy of a massive SdS black hole is less than the entropy of empty dS spacetime. So, the D-bound [29] on the entropy of asymptotically dS spacetimes with positive cosmological constant is satisfied.

5. Discussion

We have shown in this paper that the counterterm generating algorithm in asymptotically AdS spacetimes [10] can be generalized to the asymptotically de Sitter case. We have explicitly computed the counterterm lagrangian and associated boundary stress-energy up to $d = 8$. The results are a straightforward analytic continuation of the anti de Sitter case. However their interpretation is somewhat less clear. While the conserved charges (2.22) can be associated with the closed surface Σ, this surface does not itself enclose anything, since the topology of the hypersurface is $\mathbb{R}\times S^{d-1}$. Consequently one cannot really consider the conserved charges as being contained within Σ. However this is not as dissimilar to the asymptotically flat and anti de Sitter cases (where $\partial/\partial t$ is timelike) as it may first appear. In those situations, the conserved quantities are defined on a given spatial slice which evolves in time. Although it is natural to think of these quantities as being contained within the hypersurface this need not be so, since anything that the observers measure are contingent only upon their choice of closed spatial surface on a given spacelike slice, and not on anything that takes place within the surface [12]. Indeed, they need not even be aware than an interior exists!

Notwithstanding such interpretive subtleties, the results are tantalizingly similar to those obtained in the AdS case, and provide further evidence for a possible dS/CFT correspondence. The terms proportional to ℓ^{d-2} in the action and in the conserved masses for even d are presumably the analogues of the Casimir energy of the dual euclidean CFT, and have been shown to be such in restrictive cases [7]. The masses are consistently negative in any dimension, and the additional contributions in even d are consistently positive. Their AdS counterparts could be obtained by a Wick rotation $\ell \to i\ell$; indeed the dS procedure is in some sense a "Wick rotation" of the AdS procedure [30].

However note that the mass parameter m can be negative; although the spacetime has singularities, these are always hidden from observers outside of the cosmological horizon. For observers located in the "lower patch" of the Penrose diagram these singularities are hidden behind a future horizon, whereas they are in behind the past horizon for observers in the "upper patch". If negative values of m are permitted then the conserved mass $\mathfrak{M}$ is always positive and greater than its value in de Sitter spacetime; moreover observers outside the cosmological horizon will never encounter the singularities. Whether or not this violates the conjecture of ref. [2] will depend upon a clarification of the notion of a cosmological singularity in this context.

We also found that the counterterm lagrangian cannot always cancel divergences in the action. Specifically, for de Sitter spacetime there are divergences in the action for even values of d when the boundary geometry is S^d. These divergences are the de Sitter analogues of those found in the AdS case [11] for compact boundary geometries of the form S^d or H^d, where the latter is a compact hyperbolic space of non-trivial topology. For reasons similar to the AdS case, this need not be fatal to a putative dS/CFT correspondence conjecture — the linear divergence could be reflective of a UV divergence in the euclidean CFT.

Finally, we constructed a provisional definition of entropy by generalizing the Gibbs-Duhem relation in asymptotically flat and AdS spacetimes. By analytically continuing the t-coordinate to imaginary values *outside* of the cosmological horizon, we find that the resultant metric will have conical singularities unless the imaginary t-coordinate is appropriately periodically identified. Although the choice of periodicity needs to be more fully justified, it yields finite and well-defined values for the provisional entropy (4.20) which are in agreement with those obtained from CFT methods in $d = 2$ and which satisfy the D-bound on the entropy of asymptotically dS spacetimes. This suggests that a well-defined notion of gravitational entropy outside of a cosmological horizon can be meaningfully constructed. Its full meaning within the context of gravitational thermodynamics remains a subject for future investigation.

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Erratum

Several typographical errors are present in some of the formulae. The relations (4.12) should read

$$
\begin{aligned}
I^{d=2} &= -\frac{\left(m - 1/2 + \tau_+^2/\ell^2\right) V_1^t}{8\pi G} \\
I^{d=3} &= -\frac{\left(m + \tau_+^3/\ell^2\right) V_2^t}{8\pi G} \\
I^{d=4} &= -\frac{\left(m - 3\ell^2/8 + \tau_+^4/\ell^2\right) V_3^t}{8\pi G} \\
I^{d=5} &= -\frac{\left(m + \tau_+^5/\ell^2\right) V_4^t}{8\pi G} \\
I^{d=6} &= -\frac{\left(m - 5\ell^4/16 + \tau_+^6/\ell^2\right) V_5^t}{8\pi G} \\
I^{d=7} &= -\frac{\left(m + \tau_+^7/\ell^2\right) V_6^t}{8\pi G} \\
I^{d=8} &= -\frac{\left(m - 35\ell^6/128 + \tau_+^8/\ell^2\right) V_7^t}{8\pi G} \,.
\end{aligned}
$$

In the formulae (4.18), the factor $8\pi G$ must be replaced by $16\pi G$; these should therefore read

$$
\begin{aligned}
\mathfrak{M}^{d=2} &= -\frac{V_1}{16\pi G}(2m-1)\left\{1 + \frac{1}{4\tau^2}\ell^2(2m-1) + \mathcal{O}\left(\frac{1}{\tau^4}\right)\right\} \\
\mathfrak{M}^{d=3} &= -\frac{V_2}{16\pi G}\left(4m - \frac{\ell^2}{2\tau} + \mathcal{O}\left(\frac{1}{\tau^3}\right)\right) \\
\mathfrak{M}^{d=4} &= -\frac{V_3}{16\pi G}\left(6m - \frac{3}{4}\ell^2 + \frac{3\ell^2}{64\tau^4}(8m-\ell^2)^2 + \mathcal{O}\left(\frac{1}{\tau^6}\right)\right) \\
\mathfrak{M}^{d=5} &= -\frac{V_4}{16\pi G}\left(8m - \frac{\ell^4}{2\tau} + \mathcal{O}\left(\frac{1}{\tau^3}\right)\right) \\
\mathfrak{M}^{d=6} &= -\frac{V_5}{16\pi G}\left(10m - \frac{5}{8}\ell^4 - \frac{5\ell^6}{64\tau^2} + \mathcal{O}\left(\frac{1}{\tau^6}\right)\right) \\
\mathfrak{M}^{d=7} &= -\frac{V_6}{16\pi G}\left(12m - \frac{15}{32}\frac{\ell^6}{\tau} + \mathcal{O}\left(\frac{1}{\tau^3}\right)\right) \\
\mathfrak{M}^{d=8} &= -\frac{V_7}{16\pi G}\left(14m - \frac{35}{64}\ell^6 - \frac{7}{64}\frac{\ell^8}{\tau^2} + \mathcal{O}\left(\frac{1}{\tau^4}\right)\right) .
\end{aligned}
$$

There is a typo in formula (4.19), so it must read as

$$
\beta_H = \left|\frac{4\pi}{(-N'(r))}\right|_{r=\tau_+} = \left|\frac{-f'(\tau)}{4\pi f^2(\tau)}\right|^{-1}_{\tau=\tau_+} .
$$

Finally relation (4.21) should read

$$
S_d = \frac{\left(\tau_+^d - (d-2)m\ell^2\right)\beta_H V_{d-1}}{8\pi G\ell^2} \,.
$$